

# HILBERT SCHEMES OF RATIONAL CURVES ON FANO HYPERSURFACES

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**Abstract.** In this paper we try to further explore the linear model of the moduli of rational maps. Our attempt yields following results. Let  $X \subset \mathbf{P}^n$  be a generic hypersurface of degree  $h$ . Let  $R_d(X, h)$  denote the open set of the Hilbert scheme parameterizing irreducible rational curves of degree  $d$  on  $X$ . We obtain that

- (1) If  $4 \leq h \leq n-1$ ,  $R_d(X, h)$  is an integral, local complete intersection of dimension
- $$(0.1) \quad (n+1-h)d+n-4.$$
- (2) If furthermore  $(h^2-n)d+h \leq 0$  and  $h \geq 4$ , in addition to part (1),  $R_d(X, h)$  is also rationally connected.

## 1 Introduction

We work over the field  $\mathbf{C}$  throughout. Hypersurfaces  $X$  of projective space  $\mathbf{P}^n$  can be classified into three different categories: (1) Fano, (2) Calabi-Yau, (3) of general type. In our previous papers [11], [12], [13], we conclude that, in all three categories, the normal sheaves of rational curves on general hypersurfaces have vanishing higher cohomology groups. This property is local. In this paper, we concentrate on the global properties in first category, Fano hypersurfaces. In Fano case, we expect that the “parameter” spaces of rational curves on  $X$  has a positive dimension, and so there are plenty of rational curves that all have no obstruction. First let’s state the main theorem.

Let  $\mathbf{P}^n$  be projective space of dimension  $n$  over complex numbers  $\mathbb{C}$ . Let

$$(1.1) \quad R_d(X, h) \subset \{c : c \subset X\}$$

denote the open set of the Hilbert scheme parameterizing irreducible rational curves of degree  $d$  on  $X$ . This is a subscheme of the Hilbert scheme of rational curves of degree  $d$  in  $\mathbf{P}^n$ .

**THEOREM 1.1.** (*Main theorem*).

(a) If  $4 \leq h \leq n-1$ ,  $R_d(X, h)$  is an integral, local complete intersection of dimension

$$(1.2) \quad (n+1-h)d+n-4.$$

(b) Furthermore, if  $(h^2-n)d+h \leq 0$  and  $h \geq 4$ ,  $R_d(X, h)$  is a rationally connected, integral, local complete intersection of dimension

$$(1.3) \quad (n+1-h)d+n-4.$$

**Remark 1.1.** A generic member of  $R_d(X, h)$  in the range of theorem 1.1 is a smooth rational curve. This assertion is not included in the theorem. As far as

we know, there are elementary proofs of the existence for lines, but there are no elementary proof for a general degree  $d$ . Clemens' proof on quintic threefolds ([2]) is one of them we know for general degree  $d$ . The importance of this remark is that in the range  $4 \leq h \leq n-1$ , J. Harris, M. Roth and J. Starr's  $R_d(X)$  is a non-empty open set of our  $R_d(X, h)$  because  $R_d(X, h)$  will be proved to be irreducible.

### 1.1 Related work

(1) The results in our previous papers [11], [12], [13] imply that  $R_d(X, h)$  for  $n \geq 4$  is a reduced, local complete intersection of dimension

$$(n+1-h)d+n-4,$$

where the negative  $(n+1-h)d+n-4$  is interpreted as the Hilbert scheme being empty, and  $X$  could be a generic complete intersection. The main theorem in this paper only addresses the remaining parts, the irreducibility and rational connectivity of  $R_d(X, h)$ . It is clear that the irreducibility could only occur when  $h$  is relatively small.

(2) Main theorem is an extension of results in two papers [6], [7] by J. Harris, J. Starr et al, in which they initiated the study of the open set  $R_d(X)$  of the Hilbert scheme parameterizing smooth rational curves of degree  $d$ . This is achieved through a detailed analysis of Kontsevich's moduli spaces of stable maps. They are followed by the works of Beheshti, Kumar and others. See section 4 for the details.

(3) Our method is different from that in [6], [7]. This difference stems from the beginning choice of "parameter" spaces of rational maps, i.e. a parameter space that parametrizes families of rational maps. They used the Kontsevich's moduli spaces of stable maps, we use a linear model of it. Both methods analyze the structures of "parameter spaces" that extend to the "boundaries". The difference is rooted in string theory's approach of fields in "non-linear model" versus "linear model".

### 1.2 Outline of the proof

In string theory, there are two different theories, "non linear sigma model" and "gauged linear sigma model". Kontsevich's moduli space of stable maps is a starting point of the rigorous, mathematical theory for "non linear sigma model". Our research focus on the mathematical structures of fields in "gauged linear sigma model", which is also called a linear model of stable moduli in [4]. There is a filtration on this model which is helplessly simple on its own. However its interplay with hypersurfaces is non trivial. The reason to use the linear model is that, the incidence scheme of rational maps on generic hypersurface in the case of study, is a "mostly" smooth subscheme of a projective space. Once the scheme is smooth, everything else will follow automatically. The linear model has advantages and disadvantages when comparing with Kontsevich's moduli space of stable maps. Our general idea in [11], [12], [13] and this paper is to re-organize local coordinates of the linear model by breaking it down to finite blocks, then to analyze each block one-by-one. This is accomplished in section 2. This method will turn the disadvantages of the linear model to its advantages.

Let  $S = \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$  be the space of all hypersurfaces of degree  $h$ .

Let

$$\mathbb{C}^{(n+1)(d+1)}$$

be the vector space,

$$(H^0(\mathcal{O}_{\mathbf{P}^1}(d))^{\oplus n+1}$$

whose open subset parametrizes the set of maps

$$\mathbf{P}^1 \rightarrow \mathbf{P}^n$$

whose push-forward cycles have degree  $d$ .<sup>1</sup> Throughout the paper, we let

$$M = \mathbb{C}^{(n+1)(d+1)}.$$

$M$  has affine coordinates. The “gauged linear sigma model” uses the space  $M$  that has a stratification of closed subvarieties,

$$(1.4) \quad M = M_d \supset M_{d-1} \supset \cdots \supset M_0 = \{\text{constant maps}\},$$

where

$$(1.5) \quad M_i = \{(gc_0, \dots, gc_n) : g \in H^0(\mathcal{O}_{\mathbf{P}^1}(d-i)), c_j \in H^0(\mathcal{O}_{\mathbf{P}^1}(i))\}.$$

This stratification makes it impossible to view  $M$  as a space of morphisms of the same degree  $d$ , i.e.  $M_d \neq \text{Hom}_d(\mathbf{P}^1, X)$ .

Let

$$\Gamma$$

be the incidence scheme

$$(1.6) \quad \{(c, f) \in M \times S : c^*(f) = f(c(t)) = 0\}$$

Let  $\Gamma_f$  be the projection of the fibre of  $\Gamma$  over  $f$  to  $M$ .

The natural dominant rational map,

$$(1.7) \quad \Gamma_f \xrightarrow{\mathcal{R}} R_d(X, h),$$

reduces theorem 1.1 to showing that  $\Gamma_f$  is a rationally connected, integral variety of the expected dimension. This rational map  $\mathcal{R}$  will be constructed and verified by the results in [9], I 6.6.1, II 2.7. and in [10], prop. 0.9. We’ll discuss the details of this in section 4.

Using this conversion, in the rest of the paper we concentrate on the scheme  $\Gamma_f$ . Notice  $\Gamma_f$  has an induced filtration

$$(1.8) \quad \Gamma_f \supset (M_{d-1} \cap \Gamma_f) \supset \cdots \supset (M_0 \cap \Gamma_f).$$

Notice by results in [9] (mentioned above)  $\mathcal{R}$  is regular on the inverse of  $R_d(X, h)$  because the rational curves in  $R_d(X, h)$  are all irreducible. But it may not be regular on the lower stratum of (1.4).

Then theorem 1.1 follows from the propositions on  $\Gamma_f$  below .

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<sup>1</sup>The automorphism of  $\mathbf{P}^1$  induces a  $PGL(2)$  group action on  $\mathbf{P}(\mathbb{C}^{(n+1)(d+1)})$ . Let

$$PGL(2)(c_0) \subset \mathbf{P}(\mathbb{C}^{h(d+1)})$$

be the orbit of  $c_0 \in \mathbf{P}(\mathbb{C}^{h(d+1)})$ .

PROPOSITION 1.2. *If  $4 \leq h \leq n$ , then for each  $d \geq 1$ , the scheme*

$$(1.9) \quad \Gamma_f \setminus M_0$$

*is smooth.*

**Remark 1.2.** *The scheme  $\Gamma_f$  is singular at the points in  $M_0$ .*

PROPOSITION 1.3. *If  $4 \leq h \leq n - 1$ , then for each  $d \geq 1$ , the scheme*

$$(1.10) \quad \Gamma_f \setminus M_0$$

*is connected.*

**Remark 1.3.** *When  $h = n$  our method failed to prove the connectivity of  $\Gamma_f$ .*

PROPOSITION 1.4. *If  $h \geq 4$  and  $(h^2 - n)d + h \leq 0$ , then the scheme*

$$(1.11) \quad \Gamma_f \setminus M_0$$

*is a rationally connected, integral, complete intersection of  $M$  defined by*

$$(1.12) \quad f(c(t_1)) = \cdots = f(c(t_{hd+1})) = 0,$$

*where  $t_1, \dots, t_{hd+1}$  are any distinct points of  $\mathbf{P}^1$ .*

The propositions 1.3, 1.4 follow from the proposition 1.2 which follows from a rather plausible, but difficult lemma

LEMMA 1.5. *Let  $\mathcal{G}$  be the Gauss map*

$$(1.13) \quad X \rightarrow (\mathbf{P}^n)^*.$$

*Let  $c : \mathbf{P}^1 \rightarrow X$  be a non-constant regular map (with an image of any degree). Assume  $X$  is generic and  $h \geq 4$ . Then for generic*

$$(t_1, \dots, t_h) \in \text{Sym}^h(\mathbf{P}^1),$$

$$\mathcal{G}(c(t_1)), \dots, \mathcal{G}(c(t_h))$$

*are linearly independent.*

## 2 Smoothness of the linear model

Lemma 1.5 is the key to the results. Its proof lies in the heart of one difficult question that is essential to many important problems in this area. In this paper we would not explore this difficult question, but refer it to the complete papers [11], [12], and [13]. Let's prove lemma 1.5.

*Proof.* of lemma 1.5: We prove it by a contradiction. Suppose there are a generic hypersurface  $X_0 = \text{div}(f_0)$  of degree  $h$ , a non-constant rational map  $c_0 : \mathbf{P}^1 \rightarrow X_0$ , birational to its image, and  $h$  points

$$c_0(t_1), \dots, c_0(t_h)$$

such that

$$\mathcal{G}(c_0(t_1)), \dots, \mathcal{G}(c_0(t_h))$$

are linearly dependent. Then

$$(2.1) \quad \dim(\mathcal{G}(c_0(t_1)) \cap \dots \cap \mathcal{G}(c_0(t_h))) \geq n - h + 1$$

and for any vector  $\alpha \in \mathcal{G}(c_0(t_1)) \cap \dots \cap \mathcal{G}(c_0(t_h))$ ,

$$(2.2) \quad \frac{\partial f_0}{\partial \alpha}|_{c_0(t)} = 0,$$

for all  $t \in \mathbf{P}^1$ . This is because

$$\mathcal{G}(c_0(t_1)) \cap \dots \cap \mathcal{G}(c_0(t_h)) \subset \mathcal{G}(c_0(t))$$

for all  $t$ . Let  $\{\alpha_j, j = 1, \dots, r = n - h\}$  be a set of linearly independent vectors in

$$\mathcal{G}(c_0(t_1)) \cap \dots \cap \mathcal{G}(c_0(t_h))$$

Then  $c_0$  lies on the hypersurfaces

$$(2.3) \quad \frac{\partial f_0}{\partial \alpha_j}|_{c_0(t)} = 0, j = 1, \dots, r.$$

(Notice  $f_0, \frac{\partial f_0}{\partial \alpha_j}$  are generic in the moduli of hypersurfaces). Hence it lies on the intersection

$$(2.4) \quad Y = \cap_j \left\{ \frac{\partial f_0}{\partial \alpha_j} = 0 \right\} \cap X_0.$$

Next we are going to apply theorem 1.1 in [13]. We should elaborate the requirements for the theorem. Let's denote the sequence of hypersurfaces defining the intersection  $Y$  by

$$(2.5) \quad f_0, f_1 = \frac{\partial f_0}{\partial \alpha_1}, \dots, f_r = \frac{\partial f_0}{\partial \alpha_r}.$$

There are three requirements for the proof of theorem 1.1 of [13]:

(1) the subvariety defined by  $f_j = 0, 0 \leq j \leq r$  is smooth of dimension  $n - r - 1$  at  $c_0(\mathbf{P}^1)$ ;

(2) each  $f_j, j = 0, \dots, r$  is a generic hypersurface. This is different from the actual notion "generic complete intersection" which usually means that the point

$$(2.6) \quad (f_1, \dots, f_r) \in H^0(\mathcal{O}_{\mathbf{P}^n}(h)) \times H^0(\mathcal{O}_{\mathbf{P}^n}(h-1)) \times \dots \times H^0(\mathcal{O}_{\mathbf{P}^n}(h-1))$$

is generic.

(3) the first condition (1) says the subvariety  $Y$  at  $c_0(\mathbf{P}^1)$  is a local complete intersection. The requirement is that dimension of the local complete intersection is larger than or equal to 3.

First two conditions are satisfied because  $f_0$  is generic. By our assumption  $h \geq 4$ , we obtain that

$$(2.7) \quad \dim(Y) = h - 1 \geq 3.$$

The third condition is also satisfied. Therefore by the theorem 1.1 in [13],

$$(2.8) \quad H^1(N_{c_0/Y}) = 0.$$

Next we apply  $H^1(N_{c_0/Y}) = 0$  to deduce an inequality. First let

$$(2.9) \quad c_0^*(T_Y) = \mathcal{O}_{\mathbf{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_{\dim(Y)}).$$

Because  $H^1(N_{c_0/Y}) = 0$ ,

$$(2.10) \quad a_j \geq -1, j = 1, \dots, \dim(Y).$$

Because at least one  $a_j$  is larger than or equal to 2 ( from automorphisms of  $\mathbf{P}^1$  ), we obtain that

$$(2.11) \quad c_1(c_0^*(T_Y)) \geq -\dim(Y) + 3 = -h + 4.$$

Now we use adjunction formula to find

$$(2.12) \quad c_1(c_0^*(T_Y)) = [n + 1 - (h + (n - h)(h - 1))]d.$$

Now we apply the inequality  $h \leq n - 1$  to obtain that

$$(2.13) \quad c_1(c_0^*(T_Y)) \leq -h + 3$$

Then (2.11) becomes

$$(2.14) \quad -h + 3 \geq -h + 4.$$

This is absurd. Therefore  $c_0$  does not exist. The lemma 1.5 is proved.

□

Next we prove proposition 1.2:

*Proof.* The idea of the proof is similar to that in [11] or [12]. We are going to choose affine coordinates for  $M$  and defining equations for  $\Gamma_f$ . Then use them to calculate the Jacobian matrix of  $\Gamma_f$ . Let's start with coordinates of  $M$ . We consider a  $c_0 \in \Gamma_f \setminus M_0$ . Let  $c'_0$  be the normalization of  $c_0(\mathbf{P}^1)$ . Then lemma 1.5 holds for  $c'_0$ . Let  $t_1, t_2, \dots, t_h$  be those points in lemma 1.5., i.e.,

$$\mathcal{G}(c(t_1)), \dots, \mathcal{G}(c(t_h))$$

are linearly independent. Next we extend

$$\mathcal{G}(c(t_1)), \dots, \mathcal{G}(c(t_h))$$

to coordinates of  $\mathbf{P}^n$ . That is to choose affine coordinates

$$z_0, \dots, z_n$$

of  $\mathbb{C}^{n+1}$  such that  $\{z_i = 0\}$  for  $i = 1, \dots, h$  are exactly  $\mathcal{G}(c_0(t_i))$ . Next we choose affine coordinates for  $M$ . In each copy  $H^0(\mathcal{O}_{\mathbf{P}^1}(d))$  of  $M$ , we express

$$c_j(t) = \sum_{k=0}^d c_j^k t^k \in H^0(\mathcal{O}_{\mathbf{P}^1}(d))$$

(for  $j$ -th copy) as

$$(2.15) \quad c_j(t) = \sum_{k=0}^d \theta_j^k (t - t_j)^k.$$

where  $t_j$  for  $j = 1, \dots, h$  are the those in lemma 1.5, and  $t_j = 0$  if  $j$  is not in the interval  $[1, h]$ . The  $\theta_j^k$  are affine coordinates for  $M$ . We would like to use coordinates  $w_j^k$  satisfying (a linear transformation of  $\theta_j^k$ )

$$(2.16) \quad \begin{cases} w_j^k = \theta_j^k, & k \neq 0 \\ w_j^0 = \sum_{k=0}^d \theta_j^k (t' - t_j)^k. \end{cases}$$

where  $t'$  is a generic complex number.

Let the corresponding coordinates for the point  $c_0$  be  $\bar{\theta}_j^k$ . Next we choose defining equations of  $\Gamma_f$  at  $c_0$ . Consider the following homogeneous polynomials in  $W_j^k$ .

$$(2.17) \quad \begin{cases} f(c(t')) = \sum_{j=0}^n \epsilon_j W_j^0 \\ \frac{\partial^j f(c(t_1))}{\partial t^j} \\ \dots \\ \frac{\partial^j f(c(t_h))}{\partial t^j} \end{cases} \quad \begin{matrix} j = 1, \dots, d \\ \\ j = 1, \dots, d \end{matrix}$$

We claim that these polynomials define the scheme  $\Gamma_f$ .

To see this, we let  $c$  be a point in the scheme defined by the polynomials in (2.17). Also let

$$(2.18) \quad f(c(t)) = \sum_{i=0}^{hd} \mathcal{K}_i(c) t^i.$$

Using an automorphism of  $\mathbf{P}^1$ , we may assume  $t_1 = 0$ . Then the equations

$$\frac{\partial^j f(c(t_1))}{\partial t^j} = 0, j = 1, \dots, d$$

imply that

$$(2.19) \quad \mathcal{K}_i = 0, 1 \leq i \leq d.$$

Then  $f(c(t))$  satisfying the first set of equations

$$\frac{\partial^j f(c(t_1))}{\partial t^j} = 0, j = 1, \dots, d$$

becomes

$$(2.20) \quad f(c(t)) = \mathcal{K}_0(c) + r^d \left( \sum_{i=1}^{d(h-1)} \mathcal{K}_{d+i}(c) t^i \right).$$

Next we repeat the same process inductively for the term

$$\sum_{i=1}^{d(h-1)} \mathcal{K}_{d+i}(c) t^i$$

to obtain all  $\mathcal{K}_i = 0, i \geq 1$ . At last  $\mathcal{K}_0 = 0$  because  $f(c(t')) = 0$ . Hence  $c \in \Gamma_f$ . To prove the proposition it suffices to show that the Jacobian matrix of

$$f(c(t')) = 0, \frac{\partial^j f(c(t_1))}{\partial t^j} = 0, \frac{\partial^j f(c(t_h))}{\partial t^j} = 0$$

with respect to the variables  $w_j^k$  has full rank. (Note  $w_j^k$  are the coordinates for  $c$ ). Let  $\alpha_j^k$  be the variables for  $T_{c_0}M$  with the basis  $\frac{\partial}{\partial w_j^k}$ . Consider the subspace  $V_T$  of  $T_{c_0}M$  defined by  $\alpha_0^k = 0 = \alpha_l^k$  where  $k \neq 0$  and  $l$  is not one of  $1, \dots, h$ . Then  $T_{c_0}\Gamma_f \cap V_T$  consists of all  $\alpha \in V_T$  satisfying

$$(2.21) \quad \begin{cases} \frac{\partial f(c_0(t'))}{\partial \alpha} = 0 \\ \frac{\partial^{j+1} f(c_0(t_1))}{\partial t^j \partial \alpha} = 0 & j = 1, \dots, d \\ \dots \\ \frac{\partial^{j+1} f(c_0(t_h))}{\partial t^j \partial \alpha} = 0 & j = 1, \dots, d \end{cases}$$

We start this with  $h$  equations in (2.21) in the second derivatives. They are equivalent to the equations

$$(2.22) \quad \frac{\partial f(c_0(t_1))}{\partial \alpha_1} = \dots = \frac{\partial f(c_0(t_h))}{\partial \alpha_1} = 0$$

where

$$\alpha_1 \in \text{span}(\alpha_1^1, \dots, \alpha_h^1).$$

By the lemma 1.5, we know that

$$\frac{\partial f(c_0(t_i))}{\partial \alpha_j^1} = \delta_i^j$$

where  $\delta_i^j = 0$  for  $i \neq j$  and  $\delta_i^i \neq 0$ . Then the equations (2.22) implies  $\alpha_j^1 = 0, j = 1, \dots, h$ . Next step is to consider another  $h$  equations in third derivatives.

$$(2.23) \quad \frac{\partial^3 f(c_0(t_1))}{\partial t^2 \partial \alpha} = \dots = \frac{\partial^3 f(c_0(t_h))}{\partial t^2 \partial \alpha} = 0$$

Because  $\alpha_j^1 = 0, j = 0, \dots, n$ , we simplify (2.23) to

$$(2.24) \quad \frac{\partial f(c_0(t_1))}{\partial \alpha_1^2} = \dots = \frac{\partial f(c_0(t_h))}{\partial \alpha_h^2} = 0.$$

Then we use lemma 1.5 to obtain that

$$(2.25) \quad \alpha_1^2 = \cdots = \alpha_h^2 = 0.$$

Recursively we obtain that the solution to the system of linear equations (2.21) is all  $\alpha_j^k, j = 1, \dots, h, k = 0, \dots, d$  satisfying

$$(2.26) \quad \sum_{i=0}^n \frac{\partial f(c_0(t'))}{\partial \alpha_j^i} = 0$$

$$\alpha_j^k = 0, j = 1, \dots, h, k = 1, \dots, d.$$

This means that the set of solutions to the equations (2.21) has dimension  $h - 1$ . Thus the rank of Jacobian matrix of  $\Gamma_f$  at  $c_0$  is  $hd + 1$ , i.e. it has full rank. Hence  $\Gamma_f$  is smooth at  $c_0$  whenever  $c_0$  is a non-constant.

This completes the proof.

□

## 2.1 Connectivity and a version of “bend and break”

This section will prove proposition 1.3. In last section we proved that

$$\Gamma_f \setminus M_0$$

is a smooth variety of dimension

$$(n + 1)(d + 1) - (hd + 1).$$

To show it is irreducible, it suffices to show it is connected. The idea of the proof is to connect a generic point of  $\Gamma_f$  to a point in the lower stratum. Then by the induction it is connected to a point parametrizing the multiple of lines. This is our version of “bend-and-break”.<sup>2</sup> Let  $\Gamma'_f$  be an irreducible component of  $\Gamma_f$ . Assume  $d \geq 2$ .

Then

$$(2.27) \quad \dim(\Gamma'_f) = (n + 1)(d + 1) - (hd + 1) = (n + 1 - h)d + n$$

Let

$$(2.28) \quad M^{d-1} = \mathcal{O}_{\mathbf{P}^1}(d - 1)^{\oplus n+1}.$$

We should note that  $M_{d-1} \simeq \mathbb{C} \times M^{d-1}$ . It has a similar stratification

$$(2.29) \quad M^{d-1} = M_{d-1}^{d-1} \supset M_{d-2}^{d-1} \supset \cdots \supset M_0^{d-1} = \{\text{constant maps}\},$$

where

$$M_i^{d-1} = \{(gc_0, \dots, gc_n) : g \in H^0(\mathcal{O}_{\mathbf{P}^1}(d - 1 - i)), c_j \in H^0(\mathcal{O}_{\mathbf{P}^1}(i))\}.$$

Then every irreducible components of  $\Gamma'_f \cap M_{d-1}$  is isomorphic to an irreducible component of

$$(2.30) \quad \mathbb{C} \times \Gamma_f^{d-1},$$

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<sup>2</sup> Our “bend and break” fails when  $h \geq n$ . The failure is due to a potential existence of certain irreducible components. But we don’t have an example of such failure.

where  $\Gamma_f^{d-1}$  is defined to be

$$(2.31) \quad \{c \in M^{d-1} : c \subset f\},$$

and  $\mathbb{C}$  is an affine open set of  $\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^1}(1)))$ . Notice

$$(2.32) \quad \begin{aligned} \dim(\Gamma_f^{d-1} \cap M_0^{d-1}) &= d + n - 1 \\ \dim(\Gamma_f^{d-1}) &= (n + 1 - h)(d - 1) + n \end{aligned}$$

Because  $h \leq n - 1$ ,  $d \geq 2$ ,

$$(2.33) \quad \dim(\Gamma_f^{d-1}) > \dim(\Gamma_f^{d-1} \cap M_0^{d-1}).$$

The inequality (2.33) holds for every components of  $\Gamma_f^{d-1}$ . Therefore every component of  $\Gamma'_f \cap M_{d-1}$  contains a non-constant  $c$ . Thus inside smooth locus of  $\Gamma_f$ , every point is connected to a point in the lower stratum. Then by the induction it suffices to prove that the second lowest stratum  $\Gamma_f \cap (M_1 \setminus M_0)$ , which consists of all maps that correspond to lines, is connected. By the classical result of Fano variety of lines, this is correct. More precisely

$$\Gamma_f \cap M_1$$

is isomorphic to

$$\mathbb{C}^{d-1} \times \Gamma_f^1$$

where  $\Gamma_f^1$  is the same as (2.31) with  $d = 2$ , and  $\mathbb{C}^{d-1}$  is an affine open set of

$$\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^1}(d - 1))).$$

Then it suffices to prove

$$\Gamma_f^1$$

is irreducible. The image of  $\Gamma_f^1$  under the rational map  $\mathcal{R}$  is just an open set of Fano variety  $F(X)$  of lines on the generic hypersurface  $X = \{f = 0\}$ . It is connected by the classical result (see theorem 4.3, [9]). Therefore the proposition 1.3 is proved.

### 3 Rationally connectedness

*Proof.* Note

$$(3.1) \quad \mathbf{P}(M_0)$$

is a smooth subvariety of  $\mathbf{P}^{(n+1)d+n}$ , with dimension

$$d + n - 1.$$

Choose two generic planes  $V_{top}, V_{bott}$  in  $\mathbf{P}^{(n+1)d+n}$  with dimensions

$$nd - 1, n + d$$

respectively. Consider the dominant projection map

$$(3.2) \quad \Gamma_f \setminus (\Gamma_f \cap V_{top}) \rightarrow V_{bott}.$$

Because  $d \geq 2$ , the fibre's dimension is at least

$$(n - h)d - 1$$

which is larger than or equal to one. By Bertini's theorem, the generic fibre is a smooth complete intersection of  $hd + 1$  hypersurfaces of degree  $h$  followed by  $n + d$  many hyperplanes in a projective space of dimension  $(n + 1)d + n$ .

Notice the generic fibre satisfies

$$(3.3) \quad h(hd + 1) + n + d \leq (n + 1)d + n.$$

(because  $(h^2 - n)d + h \leq 0$ ), where the left hand side is the sum of the degrees of all hypersurfaces and right hand side is the dimension of the projective space. Hence the generic fibre is a smooth Fano variety. By V (2.1) and (2.13) of [9], it is rationally connected. By corollary 1.3, [5],

$$(3.4) \quad \Gamma_f \setminus (\Gamma_f \cap V_{top})$$

must be rationally connected. The proof is completed.

□

To summarize it, we just proved that

**THEOREM 3.1.**

(1) If  $4 \leq h \leq n - 1$ ,  $\Gamma_f$  is an integral, complete intersection of dimension

$$(3.5) \quad (n + 1 - h)d + n.$$

(2) Furthermore, if  $(h^2 - n)d + h \leq 0$  and  $h \geq 4$ ,  $\Gamma_f$  is a rationally connected, integral, complete intersection of dimension

$$(3.6) \quad (n + 1 - h)d + n.$$

*Proof.* of theorem 1.1.: Next we show that the results of theorem 3.1 also hold for the open set  $R_d(X, h)$  of Hilbert scheme. Let  $c_{bi} \in \Gamma_f$  be a point of  $\Gamma_f$  such that  $c_{bi}, \mathbf{P}^1 \rightarrow X$  is birational to its image. There is a rational map map

$$(3.7) \quad \begin{array}{ccc} \Gamma_f & \xrightarrow{\mathcal{R}_1} & Hom_{bir}(X)^{sn} \\ (c_0, \dots, c_n) & \rightarrow & graph(\{t\} \rightarrow [c_0(t), \dots, c_n(t)]) \end{array}$$

where  $sn$  stands for semi-normalization and  $\mathcal{R}$  is an local isomorphism at  $c_{bi}$ . Next we use the results from [9], namely I theorem 6.3, II comment 2.7. to construct the composition in a neighborhood of  $c_{bi}$ ,

$$(3.8) \quad Hom_{bir}(X)^{sn} \rightarrow CH(W) \rightarrow Hilb(X)^{sn}$$

Finally  $\mathcal{R}$  is defined to be the composition in a neighborhood

$$(3.9) \quad \Gamma_f \xrightarrow{\mathcal{R}_1} \text{Hom}_{\text{bir}}(X)^{sn} \xrightarrow{\mathcal{R}_2} CH(W) \xrightarrow{\mathcal{R}_3} \text{Hilb}(X)^{sn}.$$

By proposition 1.3,  $c_{bi}$  is a smooth point of  $\Gamma_f$ . Then  $\text{Hom}_{\text{bir}}(X)^{sn}$  is normal at  $c_{bi}$ . Then the map  $\mathcal{R}$  is regular at  $c_{bi}$  because  $\mathcal{R}_3$  is an isomorphism by I theorem 6.3, [9],  $\mathcal{R}_1$  is a smooth map with the fibres of dimension 1 by the argument for (2.15), [11], and  $\mathcal{R}_2$  is the projection of a fibre product with fibres of dimension 3 by II 2.7, [9], prop. 0.9 [10]. Then theorem 1.1 follows from theorem 3.1

□

## 4 Work of Harris et al.

Our main theorem extends the current known results in this area. Let  $R_d(X)$  be the open set of the Hilbert scheme parametrizing smooth, irreducible rational curves of degree  $d$ . One should notice  $R_d(x) \neq R_d(X, h)$ . In [6] and [7], J. Harris, J. Starr et al proved that

THEOREM 4.1. (*Harris, Starr et al*).

(1) If  $h < \frac{n+1}{2}, n \geq 3$ ,  $R_d(X)$  is an integral, locally complete intersection of the expected dimension

$$(n+1-h)d+n-4.$$

(2) If furthermore  $h \leq \frac{-1+\sqrt{4n-3}}{2}$  and  $n \geq 3$ , in addition to that in part (1),  $R_d(X)$  is also rationally connected.

The part (1) which is in their first paper, is furthered by Coskun, Beheshti, Kumar and many others ([1], [3], etc). It is conjectured by Coskun and Starr ([3]) that if  $h \leq n \geq 4$ ,  $R_d(X)$  is irreducible and has the expected dimension

$$(n+1-h)d+n-4.$$

THEOREM 4.2. For  $h \geq 4$ , theorem 1.1 recovers theorem 4.1, and furthermore Coskun and Starr's conjecture is correct for  $h \leq n-1, n \geq 4$ .

*Proof.* Kim and Pandharipande proved the conjecture for  $h = 1, 2$  ([8]). The case  $h = 3$  is solved by Coskun and Starr ([3]).<sup>3</sup>

The case  $4 \leq h \leq n-1$  follows from theorem 1.1, part (1), because in the range of  $4 \leq h \leq n-1$ , the variety  $R_d(X)$  is strictly contained in  $R_d(X, h)$ .

Because of the strict containment  $R_d(X) \subsetneq R_d(X, h)$ , theorem 4.1 follows from theorem 1.1.

□

**Remark 4.1.** For  $h = n$ , the irreducibility of neither  $R_d(X, h)$  nor  $R_d(X)$  are settled. For Calabi-Yau's case  $h = n+1$ , both  $R_d(X)$  and  $R_d(X, h)$  could become

---

<sup>3</sup> Their proof provides an evidence showing that for  $h = 3$ ,  $\Gamma_f$  is not smooth at  $M_1$ , i.e. the numerical condition  $h \geq 4$  for proposition 1.2 is necessary.

reducible and  $R_d(X)$  turns out to be a strictly contained, union of irreducible components of  $R_d(X, h)$ . But there is no evidence showing Calabi-Yau's situation could even partially occur in the Fano case  $h = n$ .

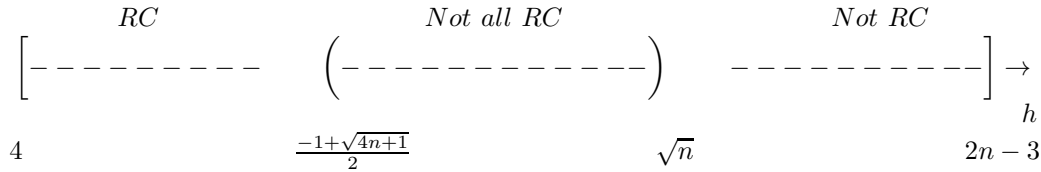
The following lemma deals with the rational connectedness.

**COROLLARY 4.3.**

(1) If  $h \leq \frac{-1+\sqrt{4n+1}}{2}$  and  $h \geq 4$ , then for each degree  $d$ , the Hilbert scheme  $R_d(X, h)$  is a rationally connected, integral, local complete intersection of the expected dimension.

(2) If  $\frac{-1+\sqrt{4n+1}}{2} < h < \sqrt{n}$  and  $h \geq 4$ , then for each degree  $d \geq \frac{h}{n-h^2}$ ,  $R_d(X, h)$  is a rationally connected, integral, local complete intersection of the expected dimension.

**Remark 4.2.** The part (1) of the corollary improves Harris and Starr's bound by a little ([7]). Part (2) reveals something new which says  $R_d(X, h)$  will not immediately become non-rationally connected as the degree of hypersurface increases. The range  $(\frac{-1+\sqrt{4n+1}}{2}, \sqrt{n})$  for  $h$  serves as a "buffer-zone" for the rational connectedness to fadeout. The following is the conjectural graph of such a distribution



where RC stands for rationally connected. In the graph this paper proves all RC statements, but did not prove any of non RC statements for which we only know a handful of indirect examples. For the irreducibility this "buffer zone" may only consist of one number. See section 5.

*Proof.* If  $h \leq \frac{-1+\sqrt{4n+1}}{2}$ ,  $h^2 + h - n \leq 0$ . Hence  $h^2 - n < 0$ . Then

$$(4.1) \quad (h^2 - n)d + h \leq 0$$

holds for all  $d \geq 1$ . Then by Main theorem 1.1, the part (1) is proved.

The part (2) of the corollary is just the part (2) of Main theorem 1.1.

□

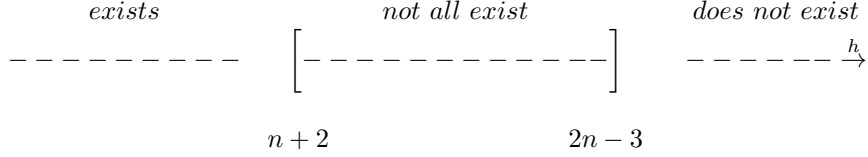
## 5 Hilbert scheme of rational curves

In this section we would like to organize our results in the area of rational curves on hypersurfaces. This extends to hypersurfaces in other two categories: Calabi-Yau, of general type. As before let  $X \subset \mathbf{P}^n$  be a generic hypersurface of degree  $h$ . Let  $R_d(X, h)$  denote the open set of the Hilbert scheme parameterizing irreducible rational curves of degree  $d$  on  $X$ .

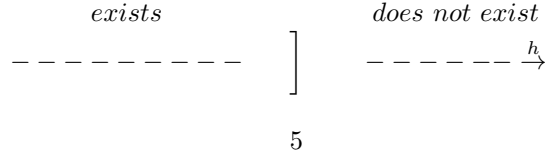
In general the full scheme structure of  $R_d(X, h)$  depends on the full scheme structure of  $X$ . But we hope that some of basic structures of  $R_d(X, h)$  may only depend on the indices  $d, h$  and  $n$ . We would like to discuss these structures. This works well in the case  $h \geq 4, n \geq 5$ . But as  $h, n$  get smaller, the situation either becomes too

simple or too complicated. They do not fit into a bigger picture which usually has a pattern of a gradual change. In the following we describe three basic structures of Hilbert scheme  $R_d(X, h)$ : (1) existence, (2) irreducibility, (3) rational connectivity.

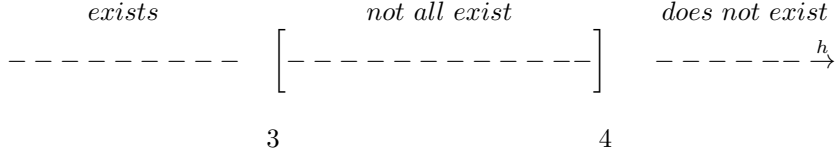
(1) For the existence, we have the following demographic picture for  $n \geq 5$ . Its correctness was proved in ([12]).



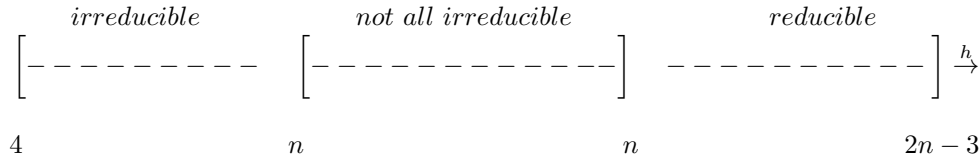
For  $n \leq 4$ , situation is subtle. If  $n = 4$ , this is the Clemens' conjecture. We have a picture (proved in [11]),



For  $n = 3$ , we have a conjectural picture



(2) For the irreducibility, we have the following demographic picture for  $h \geq 4, n \geq 5$ ,



The the statements for  $h \geq n$  are our conjectures.

(3) For the rational connectivity, we have the conjectural demographic picture in the last section.

We have skipped some of the cases for small  $h, n$  (less than 4). The situations in those cases are subtle, and some of difficult ones are already known which do not fit into the larger pictures. This means for smaller range of  $n, h$  (less than 4), moduli of hyperufaces may not be the best moduli in describing these structures.

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